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Duals of Generalized Sequence Spaces

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1. INTRODUCTION

The theory of vector-valued sequence spaces is an outgrowth of the development of (i) the theory of scalar-valued sequence spaces (to be abbreviated hereafter as SVSS), (ii) the study of Schauder decompositions in locally convex spaces and (iii) the investigation of nuclear spaces through λ-summing operators. Although the theory of vector-valued sequence spaces (hereafter abbreviated as VVSS) has been considerably developed in [1, 4, 5, 7–10, 14, 15, 17, 18], yet one finds numerous problems and their solutions to be accomplished in this theory when compared with its counterpart in SVSS. In the spirit of attempting to fulfill these gaps, we initate in this paper the study of topological and sequential duals of a VVSS $\Lambda(X)$ equipped with a locally convex topology and relate them with the generalized α-, β-, and γ- duals of $\Lambda(X)$. Indeed, we would like to investigate conditions to be laid down on a topological VVSS $\Lambda(X)$ so that its topological and sequential duals themselves behave as sequence spaces, for in that case one can conveniently develop a good deal of the duality theory between $\Lambda(X)$ and its topological dual, otherwise the situation becomes very cumbersome and, in fact, quite unpleasant. After we have evolved a technique or a process for getting these sequential and topological duals as sequence spaces, we compare the latter ones with those of the familiar α-, β-, and γ-duals of the VVSS in question. We also introduce the concept of a μ-dual of a VVSS $\Lambda(X)$, related to a given topological SVSS μ, which in particular envelops the concepts of α-, β- and γ-duals and discuss a few aspects of the duality relationship of $\Lambda(X)$ with its generalized μ-dual. A notion concerning the Köthe structure of a VVSS also appears in our discussion, namely, that of a monotone VVSS. It is weaker than the "normal" character of a VVSS and its characterization is given in Proposition 3.1. Finally, we move on to a brief

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discussion on the $\sigma\mu$-topology on a VVSS which in particular includes the notion of the normal topology $\eta(\Lambda(X), \Lambda^{\times}(Y))$ on $\Lambda(X)$. The main result in this direction is Theorem 6.7, which yields the M-character of the dual system $\langle \Lambda(X), \Lambda^{\times}(Y)\rangle$ in special circumstances (cf. Proposition 6.9).

2. Basic Concepts

Before we move on to the precise terminology and notations concerning the subject matter of the paper, let us refer to the monographs [11–13] for detailed discussions on locally convex and sequence spaces. Thus we take for granted the reader's familiarity with the rudiments of locally convex spaces and the elementary knowledge of the theory of SVSS.

Let us now assume that X and Y are two vector spaces over the same field $\mathbf{K}$ (real or complex numbers), forming a dual pair $\langle X, Y\rangle$ with respect to a bilinear form $B(x, y) = \langle x, y\rangle$, $x \in X$, and $y \in Y$. A VVSS $\Lambda(X)$ is a vector space of sequences from the space X under pointwise addition and scalar multiplication. The notations $\Phi(X)$ and $\Omega(X)$ are respectively used for the space $\{\bar{x} = \{x_i\} : x_i \in X, i \geqslant 1$ and $x_i = 0$ for all but a finite index $i\}$ and the space of all sequences from X. It is assumed throughout that a VVSS $\Lambda(X)$ will contain $\Phi(X)$ and that μ is an SVSS equipped with a Hausdorff locally convex topology T_μ generated by the family D_μ of all T_μ-continuous semi-norms.

Definition 2.1. The *generalized* μ-dual of a VVSS $\Lambda(X)$ is the space $\Lambda(X)^\mu$ defined as follows:

$$\Lambda(X)^\mu \equiv \Lambda^\mu(Y) = \{\bar{y} \in \Omega(Y) : \{\langle x_i, y_i\rangle\} \in \mu, \forall \bar{x} \text{ in } \Lambda(X)\}.$$

For $\mu = l^1$, cs, and bs (cf. [12, 13]), the corresponding μ-duals respectively known as the *generalized* α-, or *Köthe dual*, *generalized* β*-dual* and *generalized* γ*-dual*, are usually denoted by $\Lambda^{\times}(Y)$, $\Lambda^{\beta}(Y)$ and $\Lambda^{\gamma}(Y)$.

Corresponding to various permutations of the set $\mathbf{N}$ of natural numbers, we have another notion of a generalized dual of $\Lambda(X)$, known as the *generalized* δ*-dual* of $\Lambda(X)$ defined as follows:

$$\Lambda^\delta(Y) = \left\{ \bar{y} \in \Omega(Y) : \sum_{i \geqslant 1} |\langle x_i, y_{\pi(i)}\rangle| < \infty, \forall \bar{x} \in \Lambda(X); \pi \in \mathscr{P} \right\},$$

where $\mathscr{P}$ is the collection of all permutations of $\mathbf{N}$. We can similarly define $\Lambda^{\mu\mu}(X)$ and $\Lambda^{\delta\delta}(X)$, the generalized μ- and δ-duals of $\Lambda^\mu(Y)$ and $\Lambda^\delta(Y)$, respectively.

Note. The following inclusions among generalized α-, β-, γ- and δ-duals of a VVSS $\Lambda(X)$ are easily verified:

$$\Lambda^{\delta}(Y) \subset \Lambda^{\times}(Y) \subset \Lambda^{\beta}(Y) \subset \Lambda^{\gamma}(Y).$$

DEFINITION 2.2. *For a* VVSS $\Lambda(X)$ *and a subsequence* J *of* $\mathbf{N}$, *the space*

$$\Lambda_J(X) = \{\bar{x} = \{x_i\} : \text{there is a } \bar{u} = \{u_i\} \in \Lambda(X) \text{ such that } x_i = u_{n_i} \text{ for all } n_i \in J\}$$

is called the *J-step space* of $\Lambda(X)$. The *cannonical pre-image* of an element $\bar{x}_J$ in $\Lambda_J(X)$ is the sequence $\bar{x}_J$ which agrees with the coordinates of $\tilde{x}_J$ on the indices in J and is zero elsewhere. The cannonical pre-image of $\Lambda_J(X)$ is the space $\tilde{\Lambda}_J(X)$ containing cannonical pre-images of all elements of $\Lambda_J(X)$. The nth *section* of an element $\bar{x}$ in $\Omega(X)$ is the sequence $\{x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots\}$ denoted by $\bar{x}^{(n)}$. For $x \in X$ and $i \in \mathbf{N}$ *the symbol* δ_i^x is used to denote the sequence

$$\{0, 0, \ldots, \underset{i\text{th place}}{x}, 0, 0, \ldots\}.$$

For an SVSS λ and VVSS $\Lambda(X)$, we define

$$\lambda \cdot \Lambda(X) = \{\{\alpha_i x_i\} : \{\alpha_i\} \in \lambda \text{ and } \{x_i\} \in \Lambda(X)\}.$$

DEFINITION 2.3. A VVSS $\Lambda(X)$ is called (i) *normal* if $\bar{\alpha}\bar{x} \in \Lambda(X)$ whenever $\bar{x} \in \Lambda(X)$ and $\bar{\alpha} = \{\alpha_i\} \subset \mathbf{K}$ with $|\alpha_i| \leqslant 1$, for all i or equivalently $l^{\infty}\Lambda(X) \subset \Lambda(X)$; (ii) *symmetric if* $\bar{x}_{\pi} = \{x_{\pi(i)}\} \in \Lambda(X)$ whenever $\bar{x} \in \Lambda(X)$ and $\pi \in \mathscr{P}$; and (iii) *monotone* if $\Lambda(X)$ contains the cannonical pre-images of all its step spaces.

DEFINITION 2.4. A VVSS $\Lambda(X)$ is called *μ-perfect* (δ-perfect) if $\Lambda(X) = \Lambda^{\mu\mu}(X)(\Lambda(X) = \Lambda^{\delta\delta}(X))$.

Note. For $\mu = l^1$, the l^1-perfect spaces are usually referred to as *perfect.* Clearly every μ-dual of a VVSS $\Lambda(X)$ is always μ-perfect.

The space $\Lambda(X)$ is in duality with $\Lambda^{\times}(Y)$ and $\Lambda^{\beta}(Y)$ with respect to the bilinear functional G given by

$$G(\bar{x}, \bar{y}) = \sum_{i \geqslant 1} \langle x_i, y_i \rangle, \qquad \bar{x} \in \Lambda(X) \qquad \text{and} \qquad \bar{y} \in \Lambda^{\times}(Y).$$

So, we can define various polar topologies on either of the spaces $\Lambda(X)$ and $\Lambda^{\times}(Y)$ or $\Lambda^{\beta}(Y)$. The *weak, strong* and *Mackey* topologies on $\Lambda(X)$ corresponding to the dual pair $\langle \Lambda(X), \Lambda^{\times}(Y) \rangle$ are respectively denoted by $\sigma(\Lambda(X), \Lambda^{\times}(Y))$, $\beta(\Lambda(X), \Lambda^{\times}(Y))$ and $\tau(\Lambda(X), \Lambda^{\times}(Y))$. For the dual pair

$\langle \Lambda(X), \Lambda^{\times}(Y)\rangle$ there is another useful topology on $\Lambda(X)$, defined by the family $\{p_{\bar{y}}: \bar{y} \in \Lambda^{\times}(Y)\}$ of semi-norms, where

$$p_{\bar{y}}(\bar{x}) = \sum_{i \geqslant 1} |\langle x_i, y_i\rangle|, \ \bar{x} \in \Lambda(X).$$

This topology which we usually term as the *normal topology* on $\Lambda(X)$, is denoted by $\eta(\Lambda(X), \Lambda^{\times}(Y))$. Indeed, it is a special case of a more general topology to be introduced in Section 6.

Unless stated otherwise, the symbol (X, T) stands for a Hausdorff locally convex space X(l.c. TVS) equipped with a locally convex topology T generated by the family D of all T-continuous semi-norms. The topological, sequential, and algebraic duals of X are respectively denoted by X^*, X^+ and X'. Clearly $X^* \subset X^+ \subset X'$. A topological vector space X is called a *Mazur space* if $X^* = X^+$. We now recall a few definitions and results from [19].

DEFINITION 2.5. A subset B of X^+ is said to be T-limited if

$$\lim_{n \to \infty} \sup_{f \in B} |f(x_n)| = 0,$$

for every null sequence $\{x_n\}$ in X, i.e., for each sequence $\{x_n\}$ tending to zero in X. The finest locally convex topology on X having the same convergent sequences as T, is denoted by T^+ and the neighbourhood base at 0 for T^+ is the class of all balanced convex subsets u of X such that every null sequence in X is eventually contained in u.

Note. It is clear from the neighbourhood system of T^+ that $T \subset T^+$.

PROPOSITION 2.6. *For a locally convex space* (X, T),

$$(X, T^+)^+ = (X, T^+)^* = (X, T)^+.$$

Further, if (X, T) *is bornological, then* $T = T^+$.

We also need the following results respectively from [4, 5, 7, 14].

PROPOSITION 2.7. *Let* $\langle \Lambda(X), \mu(X^*)\rangle$ *be a dual pair of VVSS with* $\Lambda(X)$ *being normal and* $\Phi(X^*) \subset \mu(X^*) \subset \Lambda^{\times}(X^*)$. *Then* $\{\bar{x}^{(n)}\}$ *converges to* $\bar{x}$ *in* $\tau(\Lambda(X), \mu(X^*))$.

PROPOSITION 2.8. *If* $\mu(Y)$ *is a normal subspace of* $\Lambda^{\times}(Y)$, *then* $\sigma(\Lambda(X), \mu(Y))$ *and* $\eta(\Lambda(X), \mu(Y))$ *convergent sequences in* $\Lambda(X)$ *are the same.*

THEOREM 2.9. *Let* X *be a weakly complete locally convex space. Then* $\Lambda(X)$ *is perfect if and only if* $(\Lambda(X), \eta(\Lambda(X), \Lambda^{\times}(X^*))$ *is complete.*

PROPOSITION 2.10. *Let X be an l.c. TVS. Then the topology $\eta(\Lambda(X), \Lambda^\times(X^*))$ is compatible with the dual pair $\langle \Lambda(X), \Lambda^\times(X^*)\rangle$.*

3. GENERALIZED α-, β-, γ- AND δ-DUALS OF $\Lambda(X)$

An useful characterization of a monotone VVSS $\Lambda(X)$ in terms of the SVSS m_0, defined as the span of set of sequences formed by zeros and ones, is

PROPOSITION 3.1. *A VVSS $\Lambda(X)$, where X is a vector space, is monotone if and only if $m_0\Lambda(X) \subset \Lambda(X)$.*

Proof. Let $\Lambda(X)$ be monotone. Consider $\bar{\alpha} = \{\alpha_i\} \in m_0$ and $\bar{y} = \{y_i\} \in \Lambda(X)$. Let $\beta_1, \beta_2, \ldots, \beta_n$ be the scalars assumed by α_i's repeatedly over the respective subsequences $J_1, J_2, \ldots, J_n$ of $\mathbf{N}$ and $i_1, i_2, \ldots, i_p$ are the finite indices corresponding to distinct α_i's. Then

$$\tilde{y}_{J_i} \in \tilde{\Lambda}_{J_i}(X) \qquad \text{for} \quad i = 1, 2, \ldots, n;$$

and

$$\bar{\alpha}\bar{y} = \{\alpha_i y_i\} = \sum_{i=1}^{n} \beta_i \tilde{y}_{J_i} + \sum_{j=1}^{p} \alpha_{i_j} \delta_{i_j}^{y_{i_j}}.$$

Hence $\bar{\alpha}\bar{y} \in \Lambda(X)$ and therefore $m_0\Lambda(X) \subset \Lambda(X)$.

Conversely, let $m_0\Lambda(X) \subset \Lambda(X)$ and J be any subsequence of $\mathbf{N}$. Then for $\bar{x} \in \tilde{\Lambda}_J(X)$, we can find $\bar{y} = \{y_i\} \in \Lambda(X)$ such that $x_i = y_i$, for $i \in J$ and $x_i = 0$, for $i \in \mathbf{N} - J$. Choose $\bar{\alpha} = \{\alpha_i\} \in m_0$ such that $\alpha_i = 1$ if $i \in J$ and $\alpha_i = 0$ if $i \in \mathbf{N} - J$. Then $\bar{x} = \bar{\alpha}\bar{y}$, where $\bar{\alpha} \in m_0$ and $\bar{y} \in \Lambda(X)$. Therefore $\bar{x} \in \Lambda(X)$. This completes the proof.

Remarks. It clearly follows from the above result that a VVSS $\Lambda(X)$ is monotone if and only if $\bar{\alpha}\bar{x} \in \Lambda(X)$ whenever $\bar{\alpha} = \{\alpha_i\}$ with $\alpha_i = 0$ or 1 for all $i \geqslant 1$ and $\bar{x} \in \Lambda(X)$. Thus a normal VVSS is always monotone.

The main result of this sectionn which connects the α-dual of a VVSS $\Lambda(X)$ with its β-, γ- and δ-duals, is the following:

THEOREM 3.2. *For a dual system $\langle X, Y\rangle$ we have* (i) *$\Lambda^\times(Y) = \Lambda^\beta(Y)$ if $\Lambda(X)$ is monotone*; (ii) *$\Lambda^\times(Y) = \Lambda^\gamma(Y)$ if $\Lambda(X)$ is normal*; *and* (iii) *$\Lambda^\times(Y) = \Lambda^\delta(Y)$ if $\Lambda(X)$ is symmetric.*

Proof. It suffices to show that $\Lambda^\beta(Y) \subset \Lambda^\times(Y)$. Consider $\bar{y} = \{y_i\} \in \Lambda^\beta(Y)$. Since $m_0\Lambda(X) \subset \Lambda(X)$, it follows that $\sum_{i\geqslant 1} \langle \alpha_i x_i, y_i\rangle$

converges for every $\{\alpha_i\} \in m_0$ and $\{x_i\} \in \Lambda(X)$. Thus the series $\sum_{i\geq 1} \langle x_i, y_i\rangle$ is subseries convergent in $\mathbf{K}$ and hence $\sum_{i\geq 1} |\langle x_i, y_i\rangle| < \infty$, i.e., $\{y_i\} \in \Lambda^{\times}(Y)$.

(ii) We need prove here that $\Lambda^{\gamma}(Y) \subset \Lambda^{\times}(Y)$. Let $\bar{y} = \{y_i\}$ be a member of $\Lambda^{\gamma}(Y)$. For $\bar{x} = \{x_i\}$ in $\Lambda(X)$, choose $\{\alpha_i\} \subset \mathbf{K}$ such that $|\langle x_i, y_i\rangle| = \alpha_i \langle x_i, y_i\rangle$, $\forall i \geq 1$. Then

$$\sup_n \sum_{i=1}^{n} |\langle x_i, y_i\rangle| = \sup_n \left| \sum_{i=1}^{n} \langle \alpha_i x_i, y_i\rangle \right| < \infty$$

as $\{\alpha_i x_i\} \in \Lambda(X)$. Hence $\sum_{i\geq 1} |\langle x_i . y_i\rangle| < \infty$. Since $\bar{x}$ in $\Lambda(X)$ is arbitrary, $\bar{y} \in \Lambda^{\times}(Y)$.

(iii) The equality would follow if we could show that $\Lambda^{\times}(Y) \subset \Lambda^{\delta}(Y)$. Consider, therefore, $\bar{y} = \{y_i\}$ in $\Lambda^{\times}(Y)$. Suppose π is any permutation of $\mathbf{N}$. Then

$$\bar{x}_{\pi^{-1}} = \{x_{\pi^{-1}(i)}\} \in \Lambda(X)$$

for $\bar{x} \in \Lambda(X)$, as $\Lambda(X)$ is symmetric. Therefore

$$\sum_{i\geq 1} |\langle x_i, y_{\pi(i)}\rangle| = \sum_{i\geq 1} |\langle x_{\pi^{-1}(i)}, y_i\rangle| < \infty.$$

Hence $\{y_i\} \in \Lambda^{\delta}(Y)$. The proof is now completed.

4. Topological Duals of $\Lambda(X)$

This section is devoted to the sequential representation of the members of the topological dual of a VVSS $\Lambda(X)$ equipped with a Hausdorff locally convex topology $\mathscr{F}$. For the main result we need introduce

Definitions 4.1. (i) A VVSS $(\Lambda(X), \mathscr{F})$ is called a GK-space if the maps $P_i : \Lambda(X) \to X$, $P_i(\bar{x}) = x_i$, $i \geq 1$ and $\bar{x} = \{x_i\} \varepsilon \Lambda(X)$, are continuous. A GK-space is said to be (ii) a *GAD-space* if $\Phi(X)$ is dense in $\Lambda(X)$, i.e., $\overline{\Phi(X)} = \Lambda(X)$; (iii) a *GAK-space* if for each $\bar{x} = \{x_i\} \in \Lambda(X)$, $\bar{x}^{(n)} \to \bar{x}$ in $\mathscr{F}$. (iv) A VVSS $(\Lambda(X), \mathscr{F})$ is said to be a *GC-space* (*GSC-space*) if the maps $R_i : X \to \Lambda(X)$, $R_i(x) = \delta_i^x$, for $i \geq 1$ and $x \in X$, are continuous (sequentially continuous).

Note. When $X = \mathbf{K}$, the foregoing definitions (i) through (iii) correspond to an SVSS λ which is then called a *K-space*, an *AD-space* and an *AK-space*, respectively (cf. [3, 12]).

For a VVSS $(\Lambda(X), \mathscr{F})$ we write $[\Lambda(X)]_c = \overline{\Phi(X)}$ and $[\Lambda(X)]_s = \{\{f \circ R_i\} : f \in [\Lambda(X)]^*\}$, where $[\Lambda(X)]^*$ stands for the topological dual of $(\Lambda(X), \mathscr{F})$. Clearly $[\Lambda(X)]_s \subset \Omega(X')$. We now have

PROPOSITION 4.2. *Let X be a vector space and $\Lambda(X)$ be equipped with a Hausdorff locally convex topology $\mathscr{F}$. Then*

$$[\Lambda(X)]_s = [[\Lambda(X)]_c]_s = [\overline{\Phi(X)}]_s.$$

Proof. Since $[\Lambda(X)]^* \subset [\Lambda(X)]_c^* \equiv [[\Lambda(X)]_c]^*$, it clearly follows that

$$[\Lambda(X)]_s \subset [[\Lambda(X)]_c]_s.$$

For the over inclusion, consider $\{f \circ R_i\}$ in $[[\Lambda(X)]_c]_s$, where $f \in [\Lambda(X)]_c^*$. Applying the Hahn–Banach theorem, we get $g \in [\Lambda(X)]^*$ such that g is an extension of f. Then $\{g \circ R_i\} \in [\Lambda(X)]_s$ and $g \circ R_i = f \circ R_i$, $\forall i \geqslant 1$. Hence the result follows.

There is another way of interpreting $[\Lambda(X)]_s$ for which we need a few more definitions. Consider the dual pair $\langle \Phi(X), \Omega(X') \rangle$. For $A \subset \Phi(X)$, define

$$A^{\Omega(X')} = \{\bar{y} \in \Omega(X') : |\langle \bar{x}, \bar{y} \rangle| \leqslant 1, \forall \bar{x} \in A\}.$$

For any semi-norm p on $\Phi(X)$, let

$$A_p = \{\bar{x} \in \Phi(X) : p(\bar{x}) \leqslant 1\}.$$

Then we have

PROPOSITION 4.3. *For a VVSS $\Lambda(X)$ equipped with a locally convex topology $\mathscr{F}$ generated by the family D_Λ of semi-norms,*

$$[\Lambda(X)]_s = \bigcup \{A_p^{\Omega(X')} : p \in D_\Lambda\},$$

where

$$A_p^{\Omega(X')} \equiv (A_p)^{\Omega(X')}.$$

Proof. Let $\bar{f} \in A_p^{\Omega(X')}$ for some $p \in D_\Lambda$, $\bar{f} = \{f_i\}$, $f_i \in X'$, for $i \geqslant 1$. Then

$$|\langle \bar{x}, \bar{y} \rangle| \leqslant p(\bar{x}), \qquad \forall \bar{x} \in \Phi(X).$$

Define $G_{\bar{f}} : \Phi(X) \to \mathbf{K}$ by $G_{\bar{f}}(\bar{x}) = \langle \bar{x}, \bar{f} \rangle$. Hence $G_{\bar{f}} \in [\Phi(X)]^*$. Let $\hat{G}_{\bar{f}}$ be the unique extension of $G_{\bar{f}}$ to $\overline{\Phi(X)}$. Then $\{\hat{G}_{\bar{f}} \circ R_i\} \in [\overline{\Phi(X)}]_s = [\Lambda(X)]_s$, by Proposition 4.2. Now for $x \in X$,

$$\begin{aligned} (\hat{G}_{\bar{f}} \circ R_i)(x) &= \hat{G}_{\bar{f}}(\delta_i^x) = \langle \delta_i^x, \bar{f} \rangle = \langle x, f_i \rangle \\ &\Rightarrow \hat{G}_{\bar{f}} \circ R_i = f_i \qquad \forall i \geqslant 1. \end{aligned}$$

Hence $\bar{f} \in [\Lambda(X)]_s$ and therefore, $\bigcup \{A_p^{\Omega(X')} : p \in D_\Lambda\} \subset [\Lambda(X)]_s$.

Conversely, let $\bar{f} \in [\Lambda(X)]_s$. Then $\bar{f} = \{f \circ R_i\}$, where $f \in [\Lambda(X)]^*$. There exists $p \in D_\Lambda$ such that

$$|f(\bar{x})| \leqslant p(\bar{x}), \qquad \forall \bar{x} \in \Lambda(X).$$

Now for $\bar{x} \in \Phi(X)$, we have $\bar{x} = \sum_{i=1}^n \delta_i^{x_i}$, for some $n \in \mathbf{N}$. Therefore

$$\begin{aligned} f(\bar{x}) &= f\left(\sum_{i=1}^n \delta_i^{x_i}\right) = \sum_{i=1}^n f(\delta_i^{x_i}) = \sum_{i=1}^n (f \circ R_i)(x_i) \\ &= \langle \bar{x}, \bar{f} \rangle \\ &\Rightarrow |\langle \bar{x}, \bar{f} \rangle| \leqslant p(\bar{x}), \ \forall \bar{x} \in \Phi(X) \\ &\Rightarrow \bar{f} \in A_p^{\Omega(X')} \text{ since } |\langle \bar{x}, \bar{f} \rangle| \leqslant 1, \qquad \forall \bar{x} \in A_p. \end{aligned}$$

Consequently

$$[\Lambda(X)]_s \subset \bigcup \{A_p^{\Omega(X')} : p \in D_\Lambda\},$$

and so the result follows.

Next we have

PROPOSITION 4.4. *For a VVSS $(\Lambda(X), \mathscr{F})$, where X is a vector space, $[\Lambda(X)]_c^*$ is algebraically isomorphic to the space $[[\Lambda(X)]_c]_s$.*

Proof. Define a mapping $\Psi: [\Lambda(X)]_c^* \to [[\Lambda(X)]_c]_s$ by

$$\Psi(f) = \{f \circ R_i\}, \qquad f \in [\Lambda(X)]_c^*.$$

Then Ψ is an algebraic isomorphism from $[\Lambda(X)]_c^*$ onto $[[\Lambda(X)]_c]_s$ and the result follows.

This result immediately leads to

COROLLARY 4.5. *If $(\Lambda(X), \mathscr{F})$ is a GAD-space, then $[\Lambda(X)]^*$ is algebraically isomorphic to $[\Lambda(X)]_s$.*

Note. We observe that for a *GAD*-space $\Lambda(X)$, each member f of its topological dual can be identified with a sequence $\{f \circ R_i\}$ in $[\Lambda(X)]_s$ and vice-versa. In other words, we can treat the two spaces as the same under this identification. This identification will turn out to be very useful in proving the rest of the results of this section.

PROPOSITION 4.6. *If $(\Lambda(X), \mathscr{F})$ is a GAK-space then $[\Lambda(X)]^* \subset \Lambda^\beta(X')$.*

Proof. Let $f \in [\Lambda(X)]^*$. Then $f \equiv \{f \circ R_i\}$. Consider $\bar{x} \in \Lambda(X)$. From the hypothesis, $\bar{x}^{(n)} \to \bar{x}$ in $\mathscr{F}$ and therefore $f(\bar{x}^{(n)}) \to f(\bar{x})$. But $f(\bar{x}^{(n)}) = \sum_{i=1}^n \langle x_i, f \circ R_i \rangle$. Hence $f(\bar{x}) = \sum_{i \geqslant 1} \langle x_i, f \circ R_i \rangle$. Since $\bar{x} \in \Lambda(X)$ is arbitrary, $\{f \circ R_i\} \in \Lambda^\beta(X')$. This completes the proof.

COROLLARY 4.7. *If X is a locally convex space and $(\Lambda(X), \mathscr{F})$ is a GAK- and GC-space, then $[\Lambda(X)]^* \subset \Lambda^\beta(X^*)$.*

Restricting $(\Lambda(X), \mathscr{F})$ further, we get

PROPOSITION 4.8. *If $(\Lambda(X), \mathscr{F})$ is a barrelled, GAK- and GC-space then $[\Lambda(X)]^* = \Lambda^\beta(X^*)$.*

Proof. In view of the Corollary 4.7, we need only show that $\Lambda^\beta(X^*) \subset [\Lambda(X)]^*$. Let $\bar{f} \in \Lambda^\beta(X^*)$. Define a linear functional $F: \Lambda(X) \to \mathbf{K}$, by

$$F(\bar{x}) = \sum_{i \geqslant 1} \langle x_i, f_i \rangle.$$

Let

$$F_n(\bar{x}) = \sum_{i=1}^{n} \langle x_i, f_i \rangle \qquad \text{for} \quad n \geqslant 1.$$

Then $\{F_n\}$ is a sequence contained in $[\Lambda(X)]^*$ such that $F_n(\bar{x}) \to F(\bar{x})$ as $n \to \infty$, $\forall \bar{x} \in \Lambda(X)$. On applying the Banach–Steinhaus theorem [11, p.216], we conclude that $F \in [\Lambda(X)]^*$. Therefore $F \equiv \{F \circ R_i\}$. One can easily check that $f_i = F \circ R_i$, $i \geqslant 1$ and hence $\bar{f} \equiv F \in [\Lambda(X)]^*$.

Theorem 3.2 and Proposition 4.8 yield

COROLLARY 4.9. *Let $(\Lambda(X), \mathscr{F})$ be a barrelled, GAK- and GC-space. If* (i) *$\Lambda(X)$ is monotone, then $[\Lambda(X)]^* = \Lambda^\times(X^*)$ and* (ii) *$\Lambda(X)$ is normal, then $[\Lambda(X)]^* = \Lambda^\gamma(X^*)$.*

5. SEQUENTIAL DUALS OF $\Lambda(X)$

In this section we deal with the duality of a VVSS $\Lambda(X)$ corresponding to the dual pair $\langle X, X^+ \rangle$, where X is an l.c. TVS having its sequential dual X^+. Indeed, if

$$\Lambda^\beta(X^+) = \Big\{ \{f_i\} : f_i \in X^+, i \geqslant 1 \text{ and } \sum_{i \geqslant 1} \langle x_i, y_i \rangle \text{ converges}$$
$$\forall \bar{x} = \{x_i\} \in \Lambda(X) \Big\},$$

the pair $\langle \Lambda(X), \Lambda^\beta(X^+) \rangle$ forms a dual system. For a topological

VVSS$(\Lambda(X), \mathscr{F})$ we denote its sequential dual by $[\Lambda(X)]^+$, which can be identified algebraically with the subspace $[\Lambda(X)]_{s(+)}$ of $\Omega(X^+)$, where

$$[\Lambda(X)]_{s(+)} = \{\{f \circ R_i\} : f \in [\Lambda(X)]^+\}.$$

This correspondence between $[\Lambda(X)]_{s(+)}$ and $[\Lambda(X)]^+$ will throughout be assumed in the rest of this section without further reference.

PROPOSITION 5.1. *If $(\Lambda(X), \mathscr{F})$ is a GAK- and GSC-space, then $[\Lambda(X)]^+ \subset \Lambda^\beta(X^+)$.*

Proof. The proof follows on the lines of the proof of Proposition 4.6 and is accordingly omitted.

For the equality of $[\Lambda(X)]^+$ and $\Lambda^\beta(X^+)$, we need

LEMMA 5.2. *Let (X, T) be an l.c. TVS and f be the $\sigma(X^+, X)$-limit of a sequence $\{f_n\}$ in X^+. Then $f \in X'$.*

Proof. Straightforward.

LEMMA 5.3. *Let (X, T) be an l.c. TVS such that every $\sigma(X^+, X)$-bounded sequence in X^+ is T-limited. If f is the $\sigma(X^+, X)$-limit of a sequence $\{f_n\}$ in X^+, then $f \in X^+$.*

Proof. In view of Lemma 5.2, we need show the sequential continuity of f. So, let $\{x_n\}$ be a null sequence in X. Since the sequence $\{f_n\}$ being $\sigma(X^+, X)$-bounded, is T-limited, therefore to each $\varepsilon > 0$ we can find $n_0 \in \mathbf{N}$ such that for $n \geqslant n_0$

$$|f_i(x_n)| < \varepsilon/2, \qquad \forall i \geqslant 1.$$

Also for each $n \in \mathbf{N}$, there exists $k_0 \equiv k_0(n)$ in $\mathbf{N}$ such that

$$|f_{k_0}(x_n) - f(x_n)| < \varepsilon/2.$$

Hence for $n \geqslant n_0$, $|f(x_n)| < \varepsilon$. Thus $f(x_n) \to 0$ as $n \to \infty$ and hence $f \in X^+$.

We now come to the desired

PROPOSITION 5.4. *Let $(\Lambda(X), \mathscr{F})$ be a GAK- and GSC-space such that every $\sigma([\Lambda(X)]^+, \Lambda(X))$-bounded sequence in $[\Lambda(X)]^+$ is $\mathscr{F}$-limited. Then*

$$[\Lambda(X)]^+ = \Lambda^\beta(X^+).$$

Proof. For $\bar{g} = \{g_i\} \in \Lambda^\beta(X^+)$, define

$$F(\bar{x}) = \sum_{i \geqslant 1} \langle x_i, g_i \rangle$$

and

$$F_n(\bar{x}) = \sum_{i=1}^{n} \langle x_i, g_i \rangle, \qquad n \geqslant 1.$$

Clearly $F_n(\bar{x}) \to F(\bar{x})$ as $n \to \infty$ and $F_n \in [\Lambda(X)]^+$, for $n \geqslant 1$. Therefore by Lemma 5.3, $F \in [\Lambda(X)]^+$ and so, $F \equiv \{F \circ R_i\}$. Since one can easily verify that $F \circ R_i = g_i$, $\forall i \geqslant 1$, it follows that $\Lambda^\beta(X^+) \subset [\Lambda(X)]^+$. The result now follows from Proposition 5.1.

Concerning the relationship amongst the various duals of a VVSS $\Lambda(X)$, one has the following

PROPOSITION 5.5. *Let (X, T) be a locally convex Mazur space and $\Lambda(X)$ be normal. Assume that $\Lambda(X)$, equipped with a locally convex topology $\mathscr{F}$ compatible with the dual pair $\langle \Lambda(X), \Lambda^\times(X^*)\rangle$, is a GSC-space. Then*

$$[\Lambda(X)]^+ = [\Lambda(X)]^* = \Lambda^\times(X^*).$$

Proof. We need only show that $[\Lambda(X)]^+ \subset \Lambda^\times(X^*)$. Let $\bar{x} \in \Lambda(X)$. Then $\bar{x}^{(n)} \to \bar{x}$ in $\tau(\Lambda(X), \Lambda^\times(X^*))$ by Proposition 2.7. Hence $(\Lambda(X), \mathscr{F})$ is a *GAK*-space. Applying Proposition 5.1, we get $[\Lambda(X)]^+ \subset \Lambda^\beta(X^+) = \Lambda^\times(X^*)$. Hence the result follows.

Making use of Proposition 2.6 and the above result, we derive

COROLLARY 5.6. *If (X, T) is a bornological space and $(\Lambda(X), \mathscr{F})$ is a normal GSC-space such that $\mathscr{F}$ is compatible with the dual pair $\langle \Lambda(X), \Lambda^\times(X^*)\rangle$, then*

$$[\Lambda(X)]^+ = [\Lambda(X)]^* = \Lambda^\times(X^*).$$

COROLLARY 5.7. *Let (X, T) be a locally convex Mazur space and $(\Lambda(X), \mathscr{F})$ a normal GSC-space. Then*

$$[\sigma(\Lambda(X), \Lambda^\times(X^*)]^+ \subset \tau(\Lambda(X), \Lambda^\times(X^*)) = [\tau(\Lambda(X), \Lambda^\times(X^*)]^+$$

The preceding corollary is used in deriving the next

PROPOSITION 5.8. *Let (X, T) be a weakly complete locally convex Mazur space and $(\Lambda(X), \mathscr{F})$ a perfect, GSC-, bornological space such that the convergent sequences in $\Lambda(X)$ relative to $\mathscr{F}$ and $\sigma(\Lambda(X), \Lambda^\times(X^*))$ are the same. Then*

$$\mathscr{F} = \tau(\Lambda(X), \Lambda^\times(X^*)) = \beta(\Lambda(X), \Lambda^\times(X^*)).$$

Proof. From the hypothesis and Proposition 2.6,

$$\mathscr{F}=\mathscr{F}^{+}=[\sigma(\Lambda(X),\Lambda^{+}(X^{*}))]$$

Hence by Corollary 5.7, $\mathscr{F}$ is compatible with the dual pair $\langle\Lambda(X),\Lambda^{\times}(X^{*})\rangle$ and therefore $\mathscr{F}=\tau(\Lambda(X),\Lambda^{\times}(X^{*}))$ (see [11, p. 221]). As $(\Lambda(X),\eta(\Lambda(X),\Lambda^{\times}(X^{*})))$ is complete by Theorem 2.9 and $\eta(\Lambda(X),\Lambda^{\times}(X^{*}))$ is compatible with the dual pair $\langle\Lambda(X),\Lambda^{\times}(X^{*})\rangle$ by Proposition 2.10, it follows that $(\Lambda(X),\mathscr{F})$ is complete (cf. [11, p. 207]). Thus $(\Lambda(X),\mathscr{F})$ is a barelled space and hence

$$\mathscr{F}=\tau(\Lambda(X),\Lambda^{\times}(X^{*}))=\beta(\Lambda(X),\Lambda^{\times}(X^{*})).$$

This completes the proof.

6. $\sigma\mu$-TOPOLOGY

In general a VVSS $\Lambda(X)$ corresponding to a dual pair $\langle X,Y\rangle$ of vector spaces X and Y is not necessarily in duality with its μ-dual $\Lambda(X)^{\mu}$ where μ is an SVSS as considered in Section 2. However, we can topologize the space $\Lambda(X)$ with the help of the family D_{μ} of semi-norms generating the topology T_{μ} on μ and the members of $\Lambda(X)^{\mu}$. Indeed, for $\bar{y}=\{y_i\}\in\Lambda(X)^{\mu}$ and $p\in D_{\mu}$, define

$$p_{\bar{y}}(\bar{x})=p(\{\langle x_i,y_i\rangle\}),\qquad \text{for}\quad \bar{x}=\{x_i\}\in\Lambda(X).$$

The locally convex topology on $\Lambda(X)$ generated by the family $\{p_{\bar{y}}: p\in D_{\mu}, \bar{y}\in\Lambda(X)^{\mu}\}$ of semi-norms on $\Lambda(X)$ is denoted by $\sigma\mu(\Lambda(X),\Lambda(X)^{\mu})$ and if there is no confusion likely to arise, we will abbreviate this notation as $\sigma\mu$. Similarly we can define the $\sigma\mu$-topology on $\Lambda(X)^{\mu}$ with the help of semi-norms $\{p_{\bar{x}}: p\in D_{\mu}, \bar{x}\in\Lambda(X)\}$, where $p_{\bar{x}}(\bar{y})=p(\{\langle x_i,y_i\rangle\})$. Clearly the $\sigma\mu$-topology is the normal topology if $\mu=l^1$.

No doubt, the space $(\mu,\ T_{\mu})$ carries a good deal of impact on $(\Lambda(X),\sigma\mu)$ and to begin with we find a glimpse of this aspect contained in

PROPOSITION 6.1. *Let X be equipped with the topology $\sigma(X,Y)$ and (μ,T_{μ}) be a K-space. Then $(\Lambda(X),\sigma\mu)$ is a GK-space. Further, if (μ,T_{μ}) is an AK-space, then $(\Lambda(X),\sigma\mu)$ is a GAK-space.*

Proof. To show that $(\Lambda(X),\sigma\mu)$ is a GAK-space, consider a net $\{\bar{x}^{\alpha}\}_{\alpha\in\Delta}$ in $\Lambda(X)$ such that $\bar{x}^{\alpha}\to 0$ relative to the $\sigma\mu$-topology. Thus for $\varepsilon>0$, $p\in D_{\mu}$ and $\bar{y}\in\Lambda(X)^{\mu}$ we can find $\alpha_0\in\Delta$, $\alpha_0\equiv\alpha_0(\varepsilon,p,\bar{y})$ such that

$$p(\{\langle x_i^{\alpha},y_i\rangle\})=p_{\bar{y}}(\bar{x}^{\alpha})<\varepsilon,\quad \alpha\geqslant\alpha_0.$$

In particular, choose $\bar{y} = \delta_i^y$ for $y \in Y$. Then it follows that

$$p(\langle x_i^\alpha, y\rangle e^i) < \varepsilon, \qquad \alpha \geqslant \alpha_0;$$

where

$$e^i = \{0, 0, 0, \ldots, \underset{\substack{\downarrow \\ i\text{th place}}}{1}, 0, 0, \ldots\}.$$

Since (μ, T_μ) is a K-space, $\langle x_i^\alpha, y\rangle \to^\alpha 0$ and hence the maps

$$P_i\colon (\Lambda(X), \sigma\mu) \to (X, \sigma(X, Y)), \qquad i \geqslant 1.$$

are continuous.

For proving the GAKness of $(\Lambda(X), \sigma\mu)$, consider an arbitrary $\bar{x}$ in $\Lambda(X)$. Then for $p \in D_\mu$ and $\bar{y} \in \Lambda(X)^\mu$,

$$\begin{aligned} p_{\bar{y}}(\bar{x}^{(n)} - \bar{x}) &= p(\{\langle x_i^{(n)} - x_i, y_i\rangle\}) \\ &= p(\bar{\beta}^{(n)} - \bar{\beta}), \end{aligned}$$

where $\bar{\beta} = \{\langle x_i, y_i\rangle\} \in \mu$. As (μ, T_μ) is AK, the result follows.

Concerning the μ-perfectness of $\Lambda(X)$, we have

PROPOSITION 6.2. *Let X be equipped with the topology $\sigma(X, Y)$ and (μ, T_μ) be an AK-space. If $(\Lambda(X), \sigma\mu)$ is sequentially complete then $\Lambda(X)$ is μ-perfect, that is, $\Lambda(X) = \Lambda(X)^{\mu\mu}$.*

Proof. Let $\bar{x} \in \Lambda(X)^{\mu\mu}$. Then $\{\bar{x}^{(n)}: n \geqslant 1\} \subset \Lambda(X)$. For any $\bar{y} \in \Lambda(X)^\mu$ and $p \in D_\mu$, we have for $m < n$

$$p_{\bar{y}}(\bar{x}^{(n)} - \bar{x}^{(m)}) = p(\{0, 0, 0, \ldots, \langle x_{m+1}, y_{m+1}\rangle, \ldots, \langle x_n, y_n\rangle, 0, 0, \ldots\}).$$

But $\{\langle x_i, y_i\rangle\} \in \mu$ and as μ is AK, the last term tends to zero as $m, n \to \infty$. Hence $\{\bar{x}^{(n)}\}$ is $\sigma\mu$-Cauchy in $\Lambda(X)$. Therefore for a uniquely determined element $\bar{z}$ in $\Lambda(X)$, $\bar{x}^{(n)} \to \bar{z}$ as $n \to \infty$ in the $\sigma\mu$-topology. Thus $x_i^{(n)} \to z_i$ as $n \to \infty$ for $i \geqslant 1$ in X relative to the $\sigma(X, Y)$ topology (cf. Proposition 6.1). Since $(\bar{x}^{(n)})_i = x_i$, for all $n \geqslant i$, we have $x_i = z_i$ for $i \geqslant 1$, and so $\bar{x} = \bar{z}$. Consequently $\bar{x} \in \Lambda(X)$ and this implies that $\Lambda(X)^{\mu\mu} \subset \Lambda(X)$. Therefore $\Lambda(X)$ is μ-perfect.

Restricting X and (μ, T_μ) further, the converse of the above proposition is obtained in the following

PROPOSITION 6.3. *Let $(X, \sigma(X, Y))$ and (μ, T_μ) be (sequentially) complete such that (μ, T_μ) is also a K-space. If $\Lambda(X)$ is μ-perfect, then $\Lambda(X)$ is $\sigma\mu$-(sequentially) complete.*

Proof. We prove the result for completeness and similarly follows the other part for sequential completeness. Let therefore $\{\bar{x}^\alpha\}_{\alpha \in I}$ be a $\sigma\mu$-Cauchy net in $\Lambda(X)$. Then $\{x_i^\alpha\}_{\alpha \in I}$ is a $\sigma(X, Y)$-Cauchy net in X, for each $i \geqslant 1$ and so there exists $\bar{x} \in \Omega(X)$ such that $x_i^\alpha \to x_i$ in $\sigma(X, Y)$ for each $i \geqslant 1$. Let $\bar{y} \in \Lambda(X)^\mu$. Then $\{\langle x_i^\alpha, y_i\rangle\} \in \mu$ for each α; also

$$p(\{\langle x_i^\alpha, y_i\rangle\} - \{\langle x_i^\beta, y_i\rangle\}) = p_{\bar{y}}(\bar{x}^\alpha - \bar{x}^\beta)$$

and so $\{\langle x_i^\alpha, y_i\rangle\}_{\alpha \in I}$ is aCauchy net in μ. Hence there exists a $\{\beta_i\} \in \mu$ such that $\{\langle x_i^\alpha, y_i\rangle\} \to_\alpha \{\beta_i\}$ in T_μ. Therefore $\langle x_i^\alpha, y_i\rangle \to^\alpha \beta_i$, for $i \geqslant 1$. But $x_i^\alpha \to x_i$ in $\sigma(X, Y)$, for $i \geqslant 1$ implies that $\langle x_i^\alpha, y_i\rangle \to \langle x_i, y_i\rangle$ for each $i \geqslant 1$. Thus $\beta_i = \langle x_i, y_i\rangle$ for all $i \geqslant 1$. Hence $\{\langle x_i, y_i\rangle\} \in \mu$. As $\bar{y} \in \Lambda(X)^\mu$ is arbitrary, it follows that $\bar{x} \in \Lambda(X)^{\mu\mu} = \Lambda(X)$. But convergence of $\{\langle x_i^\alpha, y_i\rangle\}$ to $\{\beta_i\}$ in T_μ implies that $\bar{x}^\alpha \to \bar{x}$ in the space $(\Lambda(X), \sigma\mu)$. Therefore $(\Lambda(X), \sigma\mu)$ is complete.

Since the μ-dual of a VVSS $\Lambda(X)$ is always μ-perfect, the above result immediately leads to

COROLLARY 6.4. *Let (μ, T_μ) be a K-space. If $(Y, \sigma(Y, X))$ and (μ, T_μ) are (sequentially) complete, then the μ-dual $\Lambda(X)^\mu$ of a VVSS $\Lambda(X)$, equipped with the topology $\sigma\mu(\Lambda(X)^\mu, \Lambda(X))$ is also (sequentially) complete.*

Remarks. For $\mu = l^1$, Proposition 2.1, 2.2 and Theorem 2.3 of [7] are special cases of Propositions 6.2 and 6.3; whereas by virtue of Proposition 2.8, Proposition 6.3 and Corollary 6.4 include respectively the results of [5] (Proposition 2.4) and [10] (Proposition 4.3) for a suitably restricted $\Lambda(X)$.

We observe that a set A in $\Lambda(X)$ is $\sigma\mu(\Lambda(X), \Lambda(X)^\mu)$ bounded if and only if the set $A\bar{y} = \{\{\langle x_i, y_i\rangle\} : \bar{x} \in A\}$ forms a bounded subset in μ for each $\bar{y} \in \Lambda(X)^\mu$. Replacing the singletons $\{\bar{y}\}$ by a $\sigma\mu(\Lambda(X)^\mu, \Lambda(X))$-bounded subset of $\Lambda(X)^\mu$, we introduce the following

DEFINITION 6.5. A subset A of $\Lambda(X)$ is said to be *completely bounded* in $\Lambda(X)$ if $AB = \{\{\langle x_i, y_i\rangle\} : \bar{x} = \{x_i\} \in A$ and $\bar{y} = \{y_i\} \in B\}$ is a bounded subset of μ for each $\sigma\mu(\Lambda(X)^\mu, \Lambda(X))$-bounded subset B of $\Lambda(X)^\mu$.

Note. Obviously, every completely bounded subset in $\Lambda(X)$ is bounded for the $\sigma\mu$-topology on $\Lambda(X)$. For a restricted (μ, T_μ), the other implication also holds, the proof of which makes use of the following simple result proved in [4].

LEMMA 6.6. *Let (X, T) be an l.c. TVS. A subset $B \subset X$ is bounded in X if and only if for every sequence $\{x_n\} \subset B$ and $\{\alpha_i\} \subset l^1$, the sequence $\{\sum_{i=1}^n \alpha_i x_i\}$ is a Cauchy sequence in X.*

We now have the desired converse in

THEOREM 6.7. *Let $\langle X, Y\rangle$ be a dual pair such that $(Y, \sigma(Y, X))$ is sequentially complete; also let (μ, T_μ) be a sequentially complete K-space. Then every $\sigma\mu(\Lambda(X), \Lambda(X)^\mu)$-bounded subset of $\Lambda(X)$ is completely bounded.*

Proof. Let us assume that there is a $\sigma\mu(\Lambda(X), \Lambda(X)^\mu)$-bounded subset A of $\Lambda(X)$ such that A is not completely bounded. Hence we can find a $\sigma\mu(\Lambda(X)^\mu, \Lambda(X))$-bounded set B in $\Lambda(X)^\mu$ such that AB is not bounded in μ. Therefore there exists a continuous semi-norm $p \in D_\mu$ such that $p(AB)$ is not bounded in $\mathbf{K}$. Thus for each $\varepsilon > 0$, we can find $\bar{x}^1 \in A$ and $\bar{y}^1 \in B$ with

$$p(\{\langle x_i^1, y_i^1\rangle\}) \geqslant 1 + \varepsilon.$$

Since A and B are $\sigma\mu$-bounded in $\Lambda(X)$ and $\Lambda(X)^\mu$, respectively, we can find constants K_1 and M_1 satisfying

$$\sup_{\bar{x}\in A} p(\{\langle x_i, y_i^1\rangle\}) \leqslant K_1;$$

and

$$\sup_{\bar{y}\in B} p(\{\langle x_i^1, y_i\rangle) \leqslant M_1.$$

Choose $m_1 \in \mathbf{N}$ so that $2^{-m_1+1} \leqslant \varepsilon M_1^{-1}$. Then for the constants $2^{m_1}(K_1 + 2 + \varepsilon)$, we can find $\bar{x}^2 \in A$ and $\bar{y}^2 \in B$ such that

$$p(\{\langle x_i^2, y_i^2\rangle\}) \geqslant 2^{m_1}(K_1 + 2 + \varepsilon).$$

From the $\sigma\mu$-boundedness of A and B, we can find K_2 and M_2 such that

$$\sup_{\bar{x}\in A} p(\{\langle x_i, y_i^2\rangle\}) \leqslant K_2$$

and

$$\sup_{\bar{y}\in B} p(\{\langle x_i^2, y_i\rangle\}) \leqslant M_2.$$

Choose $m_2 > m_1$ so that $2^{-m_2+1} \leqslant \varepsilon M_2^{-1}$. Proceeding in this fashion, we get sequences $\{\bar{x}^n\} \subset A$ and $\{\bar{y}^n\} \subset B$, $\{K_n\}$ and $\{M_n\}$ of constants and a subsequence $\{m_n\}$ of integers with $m_{n+1} > m_n$, $m_0 = 0$ such that

$$p(\{\langle x_i^n, y_i^n\rangle\}) > 2^{m_{n-1}} \left(\sum_{i=0}^{n-1} 2^{-m_{i-1}} K_i + n + \varepsilon\right), \qquad K_0 = 0;$$

$$\sup_{\bar{x}\in A} p(\{\langle x_i, y_i^n\rangle\}) \leqslant K_n;$$

$$\sup_{\bar{y}\in B} p(\{\langle x_i^n, y_i\rangle\}) \leqslant M_n;$$

and

$$2^{-m_n+1} \leqslant \varepsilon M_n^{-1}$$

for $n = 1, 2, 3, \ldots$.

Now it follows from Lemma 6.6 that $\sum_{j=1}^{n} 2^{-m_{j-1}} \bar{y}^j$ is a $\sigma\mu(\Lambda(X)^\mu, \Lambda(X))$-Cauchy sequence in $\Lambda(X)^\mu$. Hence from Corollary 6.4, there exists a $\bar{z} \in \Lambda(X)^\mu$ with

$$\bar{z} = \sum_{j \geqslant 1} 2^{-m_{j-1}} \bar{y}^j,$$

where the limit is taken in the $\sigma\mu(\Lambda(X)^\mu, \Lambda(X))$-topology. Consider for n in $\mathbf{N}$,

$$\begin{aligned}
p_{\bar{z}}(\bar{x}^n) &= p(\{\langle x_i^n, z_i\rangle\}) \\
&\geqslant 2^{-m_{n-1}} p(\{\langle x_i^n, y_i^n\}) - p(\{\langle x_i^n, y_i^n\rangle\}) - p(\{\langle x_i^n, y_i^1\rangle\}) \\
&\quad - 2^{-m_1} p(\{\langle x_i^n, y_i^2\rangle\}) - \cdots - 2^{-m_{n-2}} p(\{\langle x_i^n, y_i^{n-1}\rangle\}) \\
&\quad - 2^{-m_n} M_n (1 + 2^{m_n - m_{n+1}} + 2^{m_n - m_{n+2}} + \cdots) \\
&\geqslant \left(\sum_{i=0}^{n-1} 2^{-m_{i-1}} K_i + n + \varepsilon \right) - K_1 - 2^{-m_1} K_2 - \cdots - 2^{-m_{n-2}} K_{n-1} - \in = n.
\end{aligned}$$

Since $\{\bar{x}^n\} \subset A$, the above inequality contradicts the $\sigma\mu$-boundedness of A. Hence the result follows.

Corresponding to a given dual system $\langle X, Y\rangle$ of vector spaces, it is not always true that the $\sigma(X, Y)$ and $\beta(X, Y)$-bounded sets are the same; however, the situation becomes pleasant when these two types of boundedness coincide. Accordingly we have [6]

Definition 6.8. A dual system $\langle X, Y\rangle$ is said to be an *M-system* if each $\sigma(X, Y)$-bounded set is $\beta(X, Y)$-bounded in X or, equivalently, each $\sigma(Y, X)$-bounded set of Y is $\beta(Y, X)$-bounded.

If a VVSS $\Lambda(X)$ is normal, the $\sigma(\Lambda^\times(Y), \Lambda(X))$-bounded and $\eta(\Lambda^\times(Y), \Lambda(X))$-bounded sets in $\Lambda^\times(Y)$ are the same by Proposition 2.8 and so concerning the dual pair $\langle \Lambda(X), \Lambda^\times(Y)\rangle$ we have

Proposition 6.9. *Let $\langle X, Y\rangle$ be a dual pair such that $(Y, \sigma(Y, X))$ is sequentially complete. If $\Lambda(X)$ is normal, then every $\eta(\Lambda(X), \Lambda^\times(Y))$-bounded set is $\beta(\Lambda(X), \Lambda^\times(Y))$-bounded; in particular, $\langle \Lambda(X), \Lambda^\times(Y)\rangle$ is an M-system.*

Proof. Immediately follows from Theorem 6.7 when $\mu = l^1$.

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